MATH2050C Selected Solutions to Assignment 2

Hand in: Section 2.4 no. 2, 8, 9, 14, 17.

Section 2.4 no. 1, 2, 7, 8, 9, 10, 14, 15, 17.

no. 2 Find inf and sup for the set $A = \{1/n - 1/m : n, m \in \mathbb{N}\}$.

Solution As $1/n - 1/m \geq -1/m \geq -1$, -1 is a lower bound for A. Furthermore, for any $\varepsilon > 0$, there is some numbers n, m (n very large and $m = 1$) satisfying $-1 + \varepsilon > 1/n - 1/m$. Therefore, -1 must be the infimum, that is, the greatest lower bound of A. Next, by a similar reasoning, 1 is the least upper bound of A.

no. 10 Find $\inf_x \sup_y h(x,y)$ and $\sup_y \inf_x h(x,y)$.

Solution $\sup_y h(x,y) = 1$, hence $\inf_x \sup_y h(x,y) = 1$. On the other hand, $\inf_x h(x,y) = 0$, so $\sup_y \inf_x h(x,y) = 0$. This example shows that the operation of taking inf and sup are not always commutative.

no. 14 Show that there is some n such that $1/2^n < y$ for any given $y > 0$.

Solution By Bernoulli's Inequality, $2^n = (1+1)^n \geq 1 + n > n$. By Archimedean Principle, there is some n such that $1/y < n$. Hence $2^n > n > 1/y$.

Supplementary Problems

1. Show for each positive number a and $n \geq 2$, there is a unique positive number b satisfying $b^n = a$. Suggestion: Use Binomial Theorem in Ex 1.

   Solution. Let $S = \{x > 0 : x^n < a\}$. Claim S is bounded from above: Pick some $N > a$ by Archimedean property, then $x^n < a$ implies $x^n < N \leq N^n$, so $N^n - x^n > 0$. By factorization $(N^{n-1} + N^{n-2}x + \cdots + x^{n-1})(N - x) > 0$. Since the first factor is positive, $N - x > 0$, that is, N is an upper bound of S. By order-completeness, $b = \sup S$ exists. Next we show that $b^n < a$ is impossible. Assume that it is true and we draw a contradiction. Letting $1 > \varepsilon > 0$ be small, we have

   $$(b+\varepsilon)^n = b^n + \sum_{k=1}^{n} \binom{n}{k} b^{n-k}\varepsilon^k = b^n + \varepsilon \sum_{k=1}^{n} \binom{n}{k} b^{n-k}\varepsilon^{k-1} \ .$$

   Using

   $$\sum_{k=1}^{n} \binom{n}{k} b^{n-k}\varepsilon^{k-1} \leq \sum_{k=1}^{n} \binom{n}{k} b^{n-k} \equiv c \ ,$$

   $(b+\varepsilon)^n \leq b^n + c\varepsilon$. If we choose ε satisfies $\varepsilon < (a - b^n)/c$, then $(b+\varepsilon)^n < b^n + c\varepsilon < a$, contradicting the fact that b is the supremum of S. A similar argument shows that $b^n > a$ is also impossible, thus leaves the only case $b^n = a$.

   Note From now on we denote this b by $a^{1/n}$, the positive n-th root of a.

2. A real number is called an algebraic number if it is a root of some equation $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ with integral coefficients. Show that the set of all algebraic numbers

is a countable set containing all rational numbers and numbers of the form $a^{1/k}, a > 0, k \geq 1$.

Solution Let A_n be the set consisting of all equations of degree equal to n, that is, $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ where $a_j \in \mathbb{Q}$. A_n is countable. Let Z_n be the set of the roots of the equations from A_n. Since each equation has at most n-many real roots, Z_n is also countable. Now the set of all algebraic numbers are $\bigcup_n Z_n$, hence it is also countable.

Note It shows that there are uncountably many transcendental numbers.